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The three-dimensional laminar asymptotic boundary layer with suction

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SUMMARY

Three-dimensional laminar flow over a flat plate with uniform suction is considered for a specific class of outer-flows. A series solution of the boundary-layer equations is obtained for the asymptotic case, that corresponding to transverse velocity which is everywhere constant. The limitations of the asymptotic solution are examined in terms of existing theory and experimental data for the analogous two-dimensional case. A numerical example, typical of aircraft flight parameters, is discussed in relation to the analogous impermeable wall case.

1. Introduction

The use of wall suction to delay transition from laminar to turbulent flow is a subject of increasing practical importance because of the implications relative to energy conservation. Solutions to the boundary-layer equations are available for several geometries, suction velocity distributions, and compressible, as well as incompressible flow. Preston [1] obtained the two-dimensional, asymptotic, constant-suction-velocity solution for a flat plate which yields the exponential velocity profile. Iglisch [2] extended the latter to provide a description of the flow field in the initial length. The limitations of the asymptotic solution relative to Iglisch's analysis are discussed by Schlichting [3]. The case of wall suction and injection prescribed by $v_0(x) \sim x^{-1/2}$, for a flat plate, was solved by Schlichting and Bussmann [4]. For uniform suction, the compressibility effect was determined by Lew and Fanucci [5]. An excellent summary of related work predating 1961 is found in Lachmann [6]. More recently, the rotating disc with suction and injection has been considered by Ackroyd [7] and the related subject of stability of flow over wings with suction has received attention from Lekondis [8] and other authors.

The problem of three-dimensional flow over a flat surface, with suction, is one that remains to be solved. Hansen and Herzig [9] obtained a similarity solution for the impermeable wall case, restricted to a specific class of outer flows, which nevertheless yielded information of physical significance. The prior work of Shanebrook [10] formulated the analogous problem for uniform suction in the context of the asymptotic solution. Such a solution will be presented here in terms of a series expansion involving straightforward application of the method of Frobenius.

The limitations of the two-dimensional (2D) asymptotic solution for a flat plate have been discussed by Schlichting [3] and White [11]. For problems of practical interest one would conclude that the initial length requirement is excessively long. A review of the experimental data upon which the pessimistic conclusion is based, notably that of Kay [12] and Head [13], reveals that the original authors' interpretation of their own data differs from that later appearing in the literature. It appears that over a range of flow parameters commonly encountered the two-dimensional exponential solution is rapidly approached. Thus the three-dimensional analog may have practical utility.

2. Formulation

Consider a flat plate with uniform suction v_0, in steady laminar, incompressible flow, as shown in Figure 1. The cartesian coordinate system chosen is such that x measures the distance in the direction of the uniform approaching stream, y is the distance normal to the plate surface, and z is perpendicular to x on the plate. Following the general outer-flow description of Hansen and Herzig [9], a) the streamlines are specified as a system of translates as shown in Figure 2 and b) the x component of outer-flow velocity, U_0, is taken as constant. Denoting by u, v, w the boundary layer velocity components in x, y, z directions, respectively, by W the outer-flow velocity in the z direction and by ν the kinematic viscosity, the governing boundary layer equations become:

Momentum:

$$u\frac{\partial u}{\partial x} + v_0\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x\text{–component}),$$

$$u\frac{\partial w}{\partial x} + v_0\frac{\partial w}{\partial y} - \nu\frac{\partial^2 w}{\partial y^2} = U_0\frac{\partial W}{\partial x} \qquad (z\text{–component});$$

Continuity:

$$\frac{\partial u}{\partial x} = 0.$$

The appropriate boundary conditions are:

$$u(x,0,z) = w(x,0,z) = w(0,y,z) = 0,$$

$$u(x,\infty,z) = U_0, \qquad W(x,\infty,z) = W(x),$$

$$v(x,0,z) = v_0 < 0,\ x > 0,$$

$$= 0,\ x < 0.$$

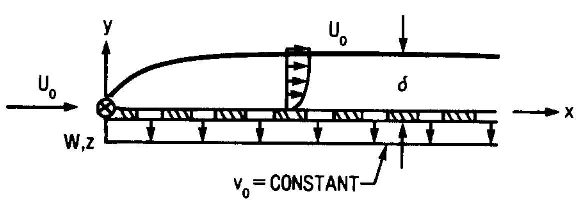

Figure 1. A flat plate with uniform suction at zero incidence in laminar flow.

The x-component of the momentum equation, the continuity equation and the applicable boundary condition suffice to describe the variation of u within the boundary layer, subject to the condition that the outer-flow velocity in the x direction is everywhere a constant. The solution for u is the well known exponential form for the asymptotic 2D boundary layer with suction, i.e.

$$u(y) = U_0(1 - e^{v_0 y/\nu}), \qquad v_0 < 0.$$

Here, note that for $v_0 > 0$ an asymptotic solution of the boundary-layer equations does not exist, either in the 2D case or this restricted 3D geometry. The case of $v_0 > 0$ is referred to as 'mass injection' in the literature and has received considerable attention because of its connection with transpiration cooling utilized in gas turbines as well as space vehicles. The subject of mass injection is not the object of this work; it suffices to state that in order to handle the problem, higher-order boundary-layer theories such as discussed by Klemp and Acrivos [14] are necessary.

With the x-component of velocity determined, it remains to solve the z-component of the momentum equation for w. The ease with which this step may be taken is stongly dependent on the form assumed for the outer-flow z-component of velocity. Hansen and Herzig found it useful to specify $W(x) = \Sigma a_i x^i$ for the case of an impermeable wall. This choice led to a similarity solution for the crossflow velocity profile in the boundary layer. However, such a result is not expected for the constant suction problem due to the invariant form u with respect to the plate surface coordinates x and z. Thus a different specification for W is sought; a convenient representation is $W(x) = \Sigma a_i e^{\alpha_i x}$, α_i any real positive integer. In this case the decomposition

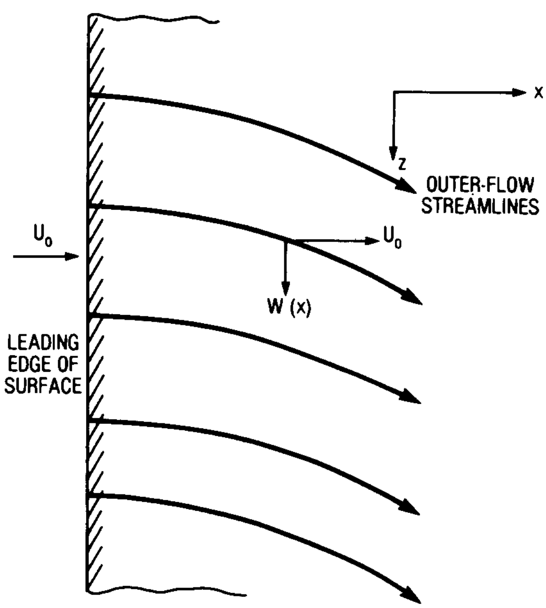

Figure 2. Streamline pattern formed by translation in z coordinate direction .

$$w = \sum_{i=1}^{n} a_i e^{\alpha_i x} f_i(y) \tag{1}$$

leads quickly to an ordinary differential equation for f_i,

$$\frac{d^2 f_i}{dy^2} + c\frac{df_i}{dy} + b(1 - e^{-cy})f_i = b, \tag{2}$$

with $c = -v_0/\nu$, $v_0 < 0$ and $b = -U_0\alpha_i/\nu$. The boundary conditions become $f_i(0) = 0$ and $f_i(\infty) = 1$.

While the second boundary condition might be expected to cause some practical difficulty, the problem is well-posed and earlier work suggested that a series solution be attempted. Before doing so, it is worthwhile to consider the question of existence of such a solution. Here it is expedient to invoke Fuchs' Theorem (cf. [15]), i.e. if, with the ordinary differential equation written in a form such that the coefficient of $d^2 f_i/dy^2$ is unity, the coefficients of the remaining terms are convergent on the interval in question, then a series solution of the form.

$$f_i = \sum_{k=0}^{\infty} A_k y^{k+s} \tag{3}$$

exists for at least the homogeneous equation. Such is definitely the case for equation (2).

3. Analysis

For convenience, the substitution $y^* = cy$ is made in equation (2). With the parameter A defined as b/c^2 and the identity

$$1 - e^{-cy} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(y^*)^n}{n!} \tag{4}$$

used to facilitate application of the method of Frobenius, equation (2) may be rewritten as follows:

$$L(f_i) = \frac{d^2 f_i}{dy^{*2}} + \frac{df_i}{dy^*} + A\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(y^*)^n}{n!} f_i = A. \tag{5}$$

Equation (5) is an ordinary second-order differential equation with an ordinary point at $x = 0$. The general solution consists of the sum of a homogeneous solution and particular solution. The complete solution of the homogeneous part will involve two arbitrary constants.

Substituting (3) in the homogeneous part of (5), expanding the term involving the parameter A, and collecting the coefficients of successive powers of y yields

$$s(s-1)A_0 y^{s-2} + [sA_0 + (s+1)sA_1]\,y^{s-1} + [(s+1)A_1 + (s+2)(s+1)A_2]\,y^{s}$$
$$+ [(s+2)A_2 + (s+3)(s+2)A_3 + A\,A_0]\,y^{s+1}$$
$$+ \left[(s+3)A_3 + (s+3)(s+4)A_4 + A\left(A_1 - \frac{A_0}{2}\right)\right] \quad y^{s+2}$$
$$+ \left[(s+4)A_4 + (s+4)(s+5)A_5 + A\left(A_2 - \frac{A_1}{2} + \frac{A_0}{6}\right)\right] y^{s+3} + \ldots$$
$$+ \left[(s+K-1)A_{K-1} + (s+K-1)(s+K)A_K\right.$$
$$\left. + A \sum_{l=1}^{K-2} \frac{(-1)^{l+K} A_{l-1}}{(K-1-l)!}\right] y + {}^{s+K-2} \ldots = 0 \qquad (6)$$

The quantities in brackets must individually vanish in order that (6) be satisfied for all values of y. This requirement applied to the term involving the lowest power of y, y^{s-1}, yields the indicial equation $s(s-1) = 0$ which has two roots, $s = 0$ and $s = 1$. Since the roots are separated by an integer, the present case is considered exceptional. This is manifested when the coefficient of y^{s-1} is examined; for $s = 0$, $sA_0 + (s+1)sA_1$ vanishes for any set of values of A_0 and A_1. Since it will be shown that a recursion relationship for the remaining coefficients A_k in terms of A_{k-1} and ultimately A_1 exists, A_0 *and* A_1 are the arbitrary constants which define the complete homogeneous solution. The choice $s = 1$ generates a series which is identical to that associated with the coefficient A_1 for $s = 0$. A general discussion of exceptional cases involving the roots of the indicial equation and the existence of series solutions is found in [16]. Noteworthy are remarks on differential equations having an ordinary rather than singular point.

In ascending powers of y, the requirement that remaining coefficients in (6) vanish yields $A_2 = -A_1/2, A_3 = A_1/6 - A\,A_0/6, A_4 = -A_1/24 - A\,(A_1 - A_0)/12$ and, in general,

$$A_k = \frac{-A_{k-1}}{k} - \frac{A}{k(k-1)} \sum_{l=1}^{k-2} \frac{A_{l-1}(-1)^{l+k}}{(k-1-l)!}, \qquad k \geqslant 3, \qquad (7)$$

with $s = 0$.

The choice $s = 2$ in the series representation (3) leads to a particular solution with coeffcieints B_k determined by the recurison relation

$$B_k = \frac{-B_{k-1}}{k+2} - \frac{A}{(k+1)(k+2)} \sum_{l=1}^{k-2} \frac{B_{l-1}(-1)^{l+k}}{(k-1-l)!} \qquad (8)$$

After collecting terms, a truncated form of the full solution may be written as:

$$f_i = (f_i)_h + (f_i)_p = \sum_{k=0}^{\infty} A_k y^{*k} + B_k y^{*k+2} = A_0 +$$

$$A_1(1 - e^{-y^*}) + AA_1\left(-\frac{y^{*4}}{12} + \frac{y^{*5}}{15} - \frac{11y^{*6}}{360} + \ldots\right)$$

$$+AA_0\left(-\frac{y^{*3}}{6} + \frac{y^{*4}}{12} - \ldots\right) + A^2A_1(\ldots) + A^2A_0(\ldots) + \ldots \qquad \text{(homogeneous)}$$

$$+A\left(\frac{y^{*2}}{2} - \frac{y^{*3}}{6} + \frac{y^{*4}}{24} - \frac{y^{*6}}{120} + \ldots\right) + A^2\left(-\frac{y^{*5}}{40} + \frac{13y^{*6}}{720} + \ldots\right) \qquad \text{(particular)}$$

$$+A^3(\ldots) + \ldots \tag{9}$$

Several features of the above solution are noteworthy. First, if the parameter A is set equal to zero, the solution is $f_i = A_0 + A_1(1 - e^{-y^*})$. The same result can be obtained directly from equation (5). The first boundary condition, $f(0) = 0$, requires that $A_0 = 0$ while the second b.c., $f(\infty) = 1$, determines the value of A_1. Such is also true of the full solution; unfortunately all terms aside from the leading one converge slowly making the task of computing the cofactor of A_1 a difficult one. Thus, while it was possible to verify the two components of the solution local to the point $y = 0$ by computing $L(f_n)$ and $L(f_p)$ and comparing with zero and A respectively, a full solution which satisfies the second boundary condition is not readily obtained.

The convergence problem is circumvented by use of the transformation

$$\zeta = e^{-cy} \tag{10}$$

Equation (2) then becomes

$$\zeta^2 \frac{d^2 f_i}{d\zeta^2} + A(1 - \zeta)f_i = A, \tag{11}$$

and the transformed boundary conditions are $f_i(1) = 0$ and $f_i(0) = 1$. The transformed equation is also representative of a class for which power-series solutions exist. The point $\zeta = 0$ corresponding to $y = \infty$ is a regular singular point so a nontrivial homogeneous solution may exist which is regular in that neighborhood. The lack of definition of $d^2f_i/d\zeta^2 |_{\zeta \to 0}$ does not present a problem insofar as the limit $d^2f_i/dy^2 |_{y \to \infty}$ is concerned. The second derivative of f with respect to y expressed in terms of $\zeta, f_i(\zeta)$ and $df_i/d\zeta$ is:

$$\frac{d^2 f_i}{dy^2} = c^2 \left[A - A(1 - \zeta)f_i(\zeta) + \zeta \frac{df_i}{d\zeta}\right]. \tag{12}$$

Thus a solution for f_i which satisfies the second boundary condition and possesses a finite first derivative at $\zeta = 0$ will give $d^2f_i/dy^2 = 0 |_{y \to \infty}$. This result is in accord with physical reasoning.

Resorting once again to the method of Frobenius, series solutions of the homogeneous and particular type are sought for equation (11) utilizing the series representation (3). The parameter s is tentatively chosen as zero for the particular solution while the indicial equation for the homogeneous solution

$$s^2 - s + A = 0 \tag{13}$$

provides two values for each choice of A:

$$s = \frac{1 \pm (1 - 4A)^{1/2}}{2}. \tag{14}$$

Note that for $A > 1/4$, both roots are complex. It turns out that for $A > 0$ the resulting velocity distribution is not of the boundary layer type, limiting choices of α_i to positive values. Since with $A < 0$ the general solution will normally involve the sum of two linearly independent solutions to the homogeneous part of (11), one of which has a leading term with an exponent less than zero, some simplification is immediately possible. The general solution must satisfy the boundary condition $f(0) = 1$ which dictates that the coefficient C_1, associated with the singular part of the solution represented by $f_i = (f_i)_h + (f_i)_p = C_1 F_1 + C_2 F_2 + (f_i)_p$, be zero.

Further limitations of the series solution related to permissible values of the parameter A motivate a discussion of the range of interest. The present analysis is applicable to incompressible fluid flow at relatively high Reynolds numbers, i.e. low kinematic viscosity. The experiments of J.M. Kay [12] provide appropriate numerical values. Kay's experiments were conducted with a free stream velocity (U_0) of 57 ft/sec., in air at approximately standard conditions of pressure and temperature; the associated kinematic viscosity is then 0.00016 ft^2/sec. The suction velocity ratio for a series of Kay's experiments was 0.0029 implying $v_0 = 0.165$ ft/sec; with a maximum value of α_i equal to 1.0 for a reasonably accurate representation of a circular arc streamline, the corresponding three-dimensional analysis would then involve $0 > A > -0.35$.

Specific choices of $A, A \leqslant 0$ lead to roots of the indicial equation which may differ by an integer. This will occur when $(1 - 4A)^{1/2}$ assumes integer values, e.g. $(1 - 4A)^{1/2} = 1,2,3,4,5$ etc. for $A = 0, -3/4, -2, -15/4, -6$ respectively. In such cases a $\ln \zeta$ term [cf. 16] may appear in general solution to (11). A solution of this type will not be developed here for two reasons: a) the largest nonzero corresponding value of A falls outside the normal range of interest, and b) in the unlikely event that the normal range is exceeded and a solution for, say $A = -2.0$ is required, a very precise approximation may be obtained from the present method by slight adjustment of the value of A. In other words, the solution for $A = -2.0$ is undoubtedly well approximated by that corresponding to $A = -1.999$ shown in Figure 3.

The recursion formula for A_k is found to be:

$$A_k = \frac{A\, A_{k-1}}{(k + s)(k + s - 1) + A}. \tag{15}$$

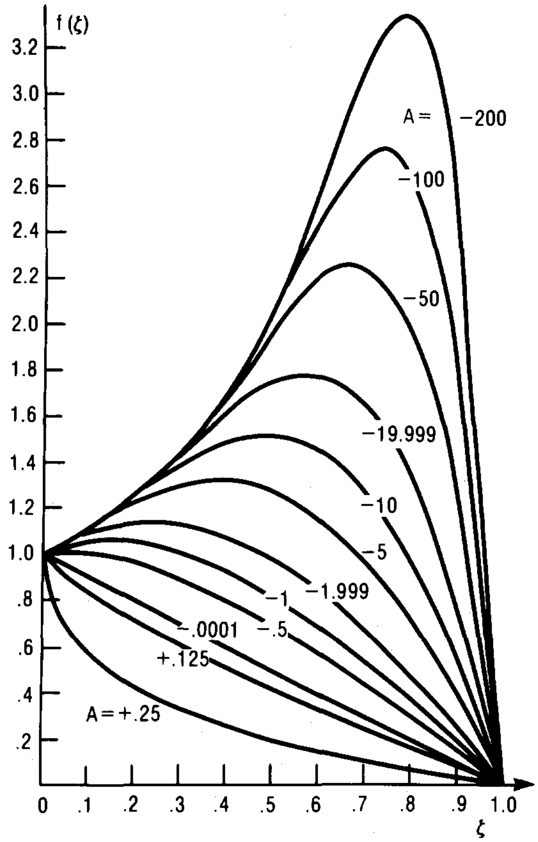

Figure 3. Solution to equation (11) for several values of the parameter A.

The recursion formula for the coefficients of the particular solution is,

$$B_k = \frac{A\,B_{k-1}}{k(k-1)+A}\,. \tag{16}$$

The earlier discussion indicates that a physically reasonable value for A is $-5/16$; the larger value of s defined by (14) is then 5/4. A corresponding complete solution to (11) in truncated form is

$$f_i = A_0\left(\zeta^{5/4} - \frac{\zeta^{9/4}}{8} + \frac{5\zeta^{13/14}}{896} - \frac{25\zeta^{19/4}}{193536} + \ldots\right)$$

$$+ 1 + \zeta - \frac{5}{27}\zeta^2 + \frac{25}{2457}\zeta^3 + \ldots \tag{17}$$

Here, in a manner similar to that encountered for the solution of equation 5, the boundary condition $f(0) = 1$ is immediately satisfied while the boundary condition $f(1) = 0$ determines the value of A_0. Note that $df_i/d\zeta|_{\zeta \to 0}$ is unity.

The complete solution is a linear combination of two infinite series; a question naturally arises as to how many terms must be carried in order to give accurate numerical values. It turns out that for A close to zero several terms suffice; as the magnitude of A increases more terms must be retained. Rather than express the remainder formally, it was found expedient to determine the number of terms required for each case by numerically calculating the values of $L(f_h)$ and $L(f_p)$, retaining enough terms so that the result was zero and A respectively to six decimal places at each point in the interval (0,1). Roughly speaking, for $0 > A > -1$, ten terms are required, $-1 > A > -20$ twenty terms and $-20 > A > -100$, forty terms. Such computation can be effectively handled on a computer of modest capability.

The form of the solution is shown in Figure 3 for several values of the parameter A. A geometrical similarity to Hansen and Herzig's solution for the impermeable wall case exists. A solution corresponding to a positive value of A is given for the sake of completeness.

4. Specification – Outer-flow

Reiterating, the cases which can be solved by the present method are those in which the outer-flow streamlines are translates and

$$U = U_0 = \text{a constant}, \; W(x) = \sum a_i e^{\alpha_i x}, \tag{18}$$

where a_i may be any real number greater than or equal to zero.

It is desired to construct outer-flow streamlines similar to the family considered by Hansen and Herzig using the above formulation. In the interest of considering physically realistic cases associated with a system of translates, the flow shall be perpendicular to the plate leading edge. A form for $z(x)$, the curve representing the outer-flow streamline, which satisfies the latter condition is

$$z(x) = de^{x/d} - e^x + c_0, \qquad c_0 \text{ a constant.}$$

A plot of $z(x)$ for several choices of d is given in Figure 4, in comparison with the circular arc streamline case. It is evident that such a choice is reasonable from geometrical viewpoint as well as meeting the requirements of the present analysis.

Turning to a more general case, the objective is to construct streamlines in accordance with the specification (18), i.e., such that

$$\frac{dz(x)}{dx} = \frac{\Sigma a_i e^{\alpha_i x}}{U_0}, \tag{19}$$

which closely approximate polynomials in x. While this can be done precisely for some special

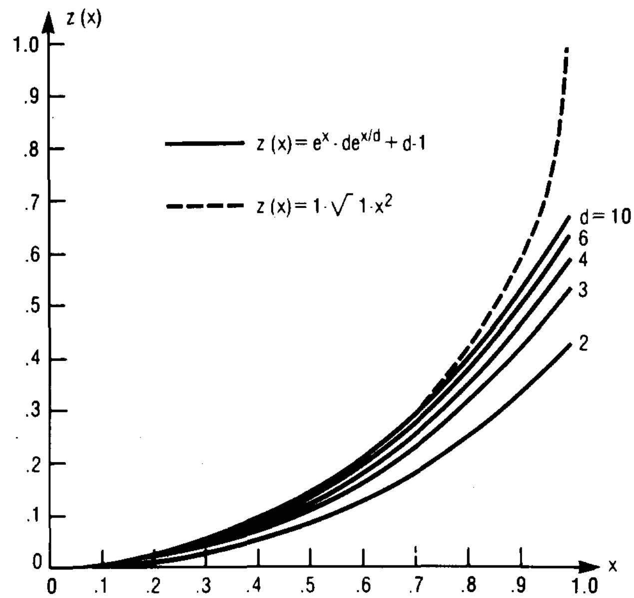

Figure 4. Simple approximation of the circular arc using a sum of exponential functions.

cases through straightforward expansion of $e^{\alpha_i x}$ in a power series, a more flexible means was sought. An adequate means of approximation is to specify a number of points $(x,z(x))$ coincident with the curve for which the representation

$$z(x) = \sum_{i=1}^{n} a_i^* e^{\alpha_i x}, \qquad a_i^* = \frac{a_i}{U_0 \alpha_i}, \tag{20}$$

is desired. The number of points chosen then fixes n. The cofactors of the argument of the exponential function, α_i are free parameters and are found through numerical experimentation. Each coordinate point $(x,z(x))$ imposes a condition on the coefficients a_i^* and thus n coordinate points results in n algebraic equations for a_i^*.

A numerical example that demonstrates the simplicity of the approach is found in the following section of this report.

The effect of suction on the crossflow velocity distribution and the boundary-layer streamline trajectories are of interest. Integration of the system

$$\frac{dy}{dx} = \frac{v_0}{U_0(1-e^{-cy})} = \frac{v}{u}, \tag{21}$$

$$\frac{dz}{dx} = \frac{\Sigma a_i e^{\alpha_i x} f_i(y)}{U_0(1-e^{-cy})} = \frac{w}{u} \tag{22}$$

determines the functions $y(x)$ and $z(x)$ corresponding to the x,y and x,z projections of the

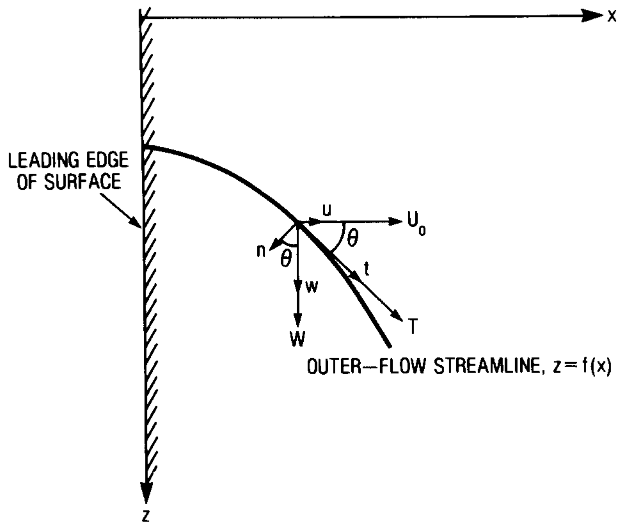

Figure 5. Resolution of velocities into components normal and tangential to outer-flow.

boundary-layer streamlines respectively. Equation (21) may be integrated explicitly but results in a transcendental equation for $y(x)$, which precludes direct integration of the second equation. Calculations of the boundary-layer streamline projections were therefore carried out numerically using a Runge-Kutta algorithm.

Boundary-layer velocity distributions were obtained in streamline coordinates, i.e., normal and tangential to the streamlines as indicated in Figure 5. The normalized form of the components, n^* and t^* are given by:

$$n^* = \frac{w - uW/U_0}{\sqrt{U_0^2 + W^2}}, \tag{23}$$

$$t^* = \frac{u + wW/U_0}{\sqrt{U_0^2 + W^2}}. \tag{24}$$

5. Numerical results

To facilitate a comparison with an impermeable-wall case solved by Hansen and Herzig, the circular-arc streamline described by $z(x) = 1 - (1 - x^2)^{1/2}$ is chosen for the outer-flow. The condition $z(0) = z^{*\prime}(0) = 0$ as well as $z(x) = z^*(x)$ at $x = 0, 0.4$ and 0.7 are imposed. Taking α_i as $i/4$, the linear algebraic system for a_i^* is

$$\begin{pmatrix} 0.25 & 0.5 & 0.75 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 1.1052 & 1.2214 & 1.3499 & 1.4918 \\ 1.1912 & 1.4191 & 1.6905 & 2.0138 \end{pmatrix} \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.0 \\ 0.0835 \\ 0.2859 \end{pmatrix}$$

Solution of this system yields

$$z(x) = 8.9550e^{1/4x} - 19.8992e^{1/2x} + 12.9333e^{3/4x} - 1.9891e^{x}.$$

The corresponding expression for $W(x)$ is obtained by differentiation and may be written as

$$W(x) = U_0(2.2388e^{1/4x} - 9.9496e^{1/2x} + 9.7e^{3/4x} - 1.9891\,e^{x}).$$

The error of these approximations is best judged by examining the accuracy of W/U_0 as a function of x. Figure 6 presents a comparison with the exact data. The error, while tolerable, could be reduced by including more terms and adopting a systematic approach in choosing the coefficients of the exponential function. However, the accuracy of the approximation is sufficient for present purposes.

A computer code was written to calculate boundary layer streamline and velocity profiles. Given values of kinematic viscosity, U_0, v_0 and a description of the outer-flow streamline, individual solutions $f_i(y)$ are computed corresponding to each cofactor of the exponential argument, α_i. The solutions are then combined in accordance with (1), to describe the boundary-layer crossflow velocity components w. Next, the velocity profiles u and w are computed at a specified number of locations x, in both the streamline and cartesian reference frames. Finally, the boundary-layer streamline for a specified initial condition is determined, the calculations carried forth until y assumes a negative value, indicating that the flow has entered the wall.

The three-dimensional uniform-suction problem is characterized by the parameters A and c. As in the two-dimensional case the boundary-layer thickness is inversely proportional to $c(\nu_0/\nu)$. The parameter $A\,(-U_0\alpha_i\nu/v_0^2)$ is related solely to the three-dimensional case and is a measure of the deviation of the crossflow profile from exponential. Note that by equation (2) $A = b = 0$, $i = 1 = N$ results in $f_N = 1 - e^{-cy}$, and therefore $W = a_N e^{\alpha_N}(1 - e^{-cy})$. The choice $A = 0$ represents a limiting case in terms kinematic viscosity, suction velocity or streamline local radius of curvature, the latter associated with α_i. For $\alpha_i = 0$ w is constant and the problem becomes two-dimensional through appropriate specification of the coordinate system.

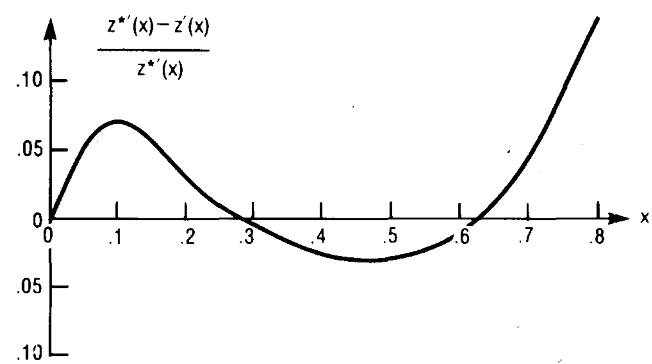

Figure 6. Measure of fit of circular-arc streamline by $\Sigma\, a_i^* e^{\alpha_i x}$

Numerical results for the circular arc streamline case are presented with $A/\alpha_i = 0.222$ and $c = 3.3 \times 10^{-4}$. Such choices are physically reasonable and result in a relatively short initial length before the asymptotic solution may be assumed to have validity. The subject of the initial-length requirement will be discussed in a following section.

Boundary layer streamlines

A comparison of boundary-layer streamlines for the impermeable wall and uniform-suction cases is somewhat confused by the differing boundary-layer thicknesses existing for the two cases. With suction $\delta \sim 4.828\ (\nu/v_0)$ while initially $\delta \sim 5.0\ (\nu x/U_0)^{1/2}$ for the impermeable wall and tends to grow faster with x than thus indicated as the crossflow develops. This fact must be kept in mind when evaluating the suction effect.

The xy and xz projections of the boundary-layer streamlines are given in Figures 7a,b respectively, for the circular-arc case. They clearly demonstrate that even at the low suction

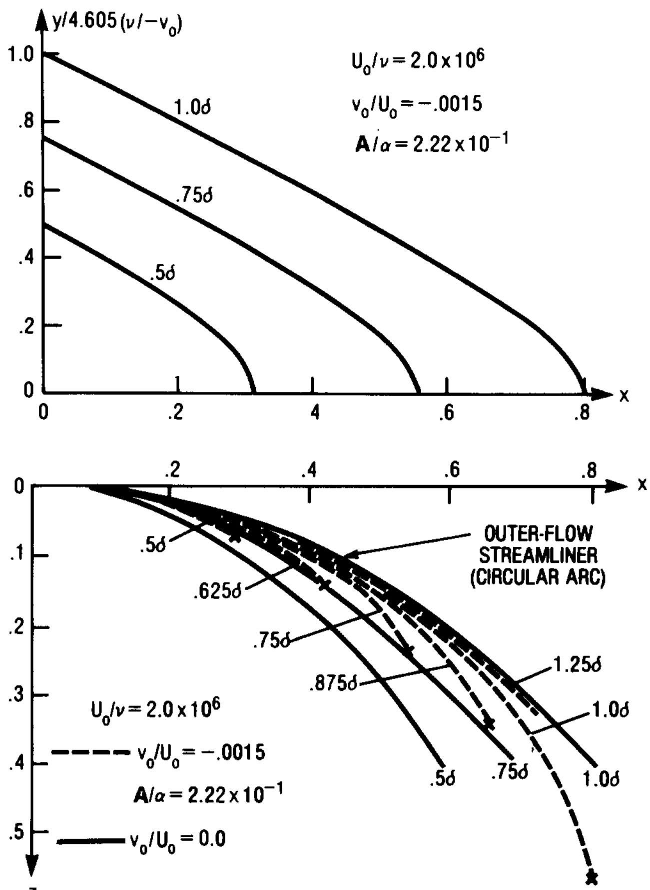

Figure 7. x-y and x-z projections of boundary-layer streamlines for a suction and the impermeable-wall case.

rates involved, fluid particles at the outer edge of the boundary layer are drawn into the wall a relatively short distance from the leading edge. The general features of the boundary-layer velocity profile shown in Figure 8 are common to both cases of suction velocity but have different influences on the streamline deflection. In the cases of the impermeable wall, fluid particles tend to stay at approximately the same distance above the plate and so the deflection of streamlines within the boundary layer is related to the development of crossflow at essentially constant height. Suction causes a vertical migration of fluid particles causing a particle in the outerflow to traverse the region of maximum crossflow component before reaching the wall. Thus the somewhat similar pattern of the boundary streamline xz projection is not surprising. A significant difference, of course, is that in the case of suction the streamlines terminate at the wall.

The effect of suction velocity on the streamline projections is shown in Figure 9a,b. Strong suction further limits the deflection of boundary-layer streamlines relative to the outer-flow streamline by decreasing the residence time of a fluid particle within the boundary layer.

Velocity profiles

The resolution of the boundary layer velocity distribution into normal and tangential components relative to a streamline is as depicted in Figure 5. Here n^* and t^* denote the normal and

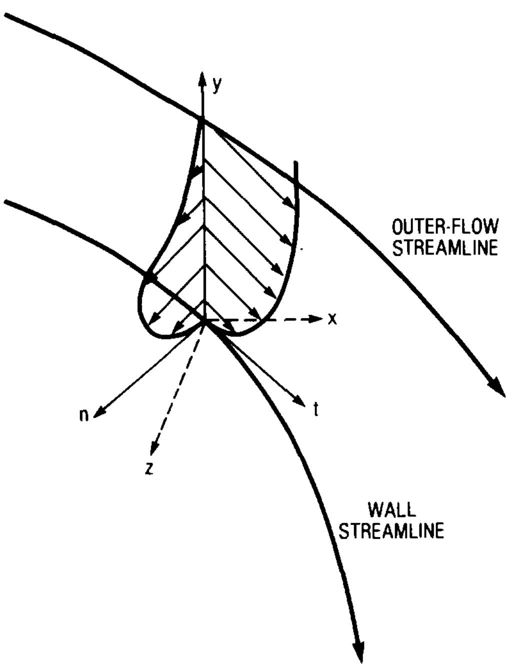

Figure 8. A typical three-dimensional boundary-layer velocity profile indicating crossflow and streamline components.

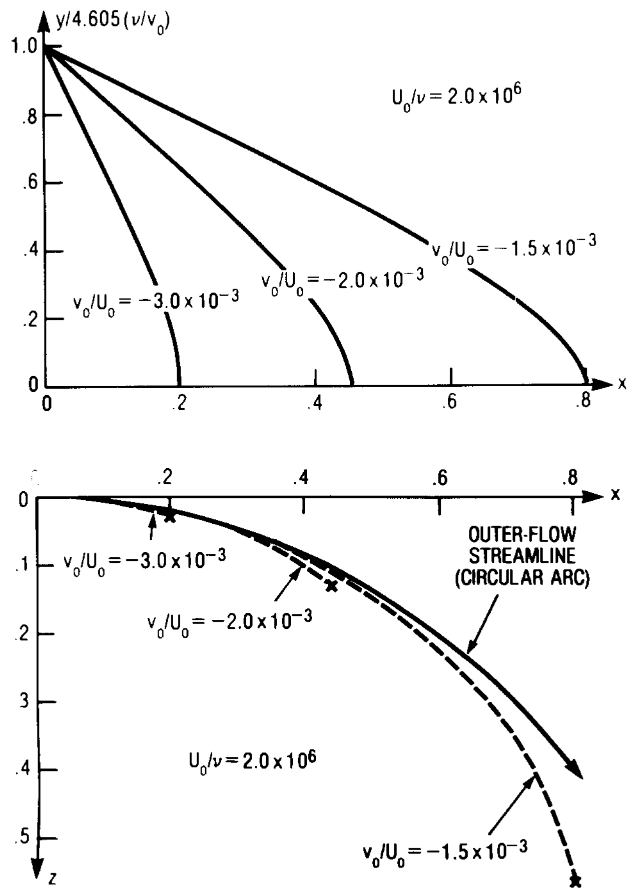

Figure 9. Boundary-layer streamline projections indicating the influence of suction.

tangential components respectively. Figure 10 shows the effect of suction on the distributions; included is the impermeable-wall case as calculated in [9]. The effect of suction in inhibiting the development of crossflow within the boundary layer is noteworthy.

6. Initial-length requirement

The condition $v(x,y,z) = v_0$, $x > 0$, v_0 a negative constant, is valid only after an initial length. As shown previously, imposing this condition decouples the two components of the momentum equation, admits the exponential distribution for $u(y)$ and leads to a relatively straightforward solution for w. Undoubtedly, the general case of uniform suction represented by

$$v(x,0,z) = v_0, \qquad x > 0, \quad v_0 < 0,$$

$$v(x,y,z) = v(x,y)$$

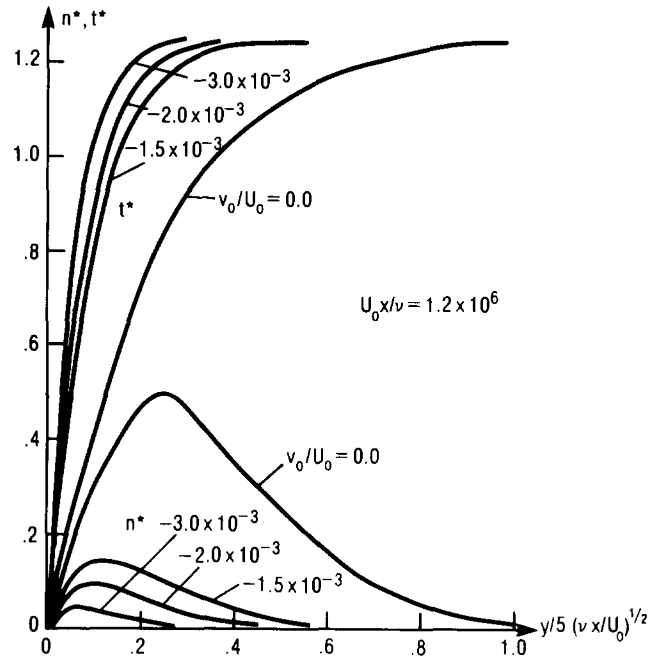

Figure 10. Boundary-layer velocity profiles in streamline coordinates indicating the influence of suction.

would be much more difficult to solve. The question remaining involves the practical importance and reality of the asymptotic solution, i.e. can the behavior described by the solution to this restricted problem be expected to occur? A complete answer is not available in the absence of experimental verification. However, insight can be gained by examining information available on the analogous two-dimensional case.

In two dimensions, a solution to the general uniform-suction case for the flat plate exists, due to Iglisch [2]. The boundary-layer velocity profile is shown to develop as a function of the dimensionless length $\xi = (-v_0/u_0)^2(U_0x/\nu)$ as indicated in Figure 11. Depending on the required degree of accuracy, a rule may be established which defines achievement of the asymptotic state in terms of a fixed value of ξ. Schlichting [3] chooses 4.0, which with regard to Figure 11 is certainly reasonable, and White [11] discusses the severe limitations of the asymptotic theory on that basis. However, an examination of experimental data applicable to the problem indicates that the results of Iglisch may be unduly pessimistic.

The experiments of J.M. Kay [12] were conducted on a 2 ft by 1 ft porous plate mounted in a closed-circuit wind tunnel with a non-porous leading-edge section of 4 inches in length. At the entrance of the test section a duct below the test plate removed the tunnel wall boundary layer. The rate of suction flow through the wall was measured by two venturi meters connecting the pump to the suction chamber. The maximum variation of mean suction velocity is reported as ten percent with a much smaller variation evident over the central portion. Data was taken under laminar and turbulent flow conditions: only the laminar flow data is of interest in the present study. Data useful in establishing the initial length requirement was taken for $v_0/U_0 = 0.0029$, $U_0 = 57$ ft/sec and was presented in the form of velocity distributions, momentum and

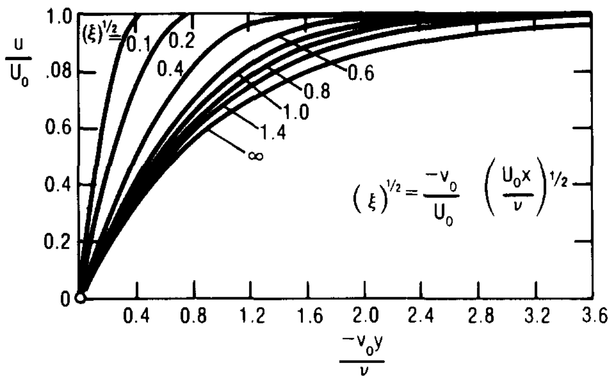

Figure 11. Velocity profiles over the initial length of a flat plate in two-dimensional flow with uniform suction. After R. Iglisch [2].

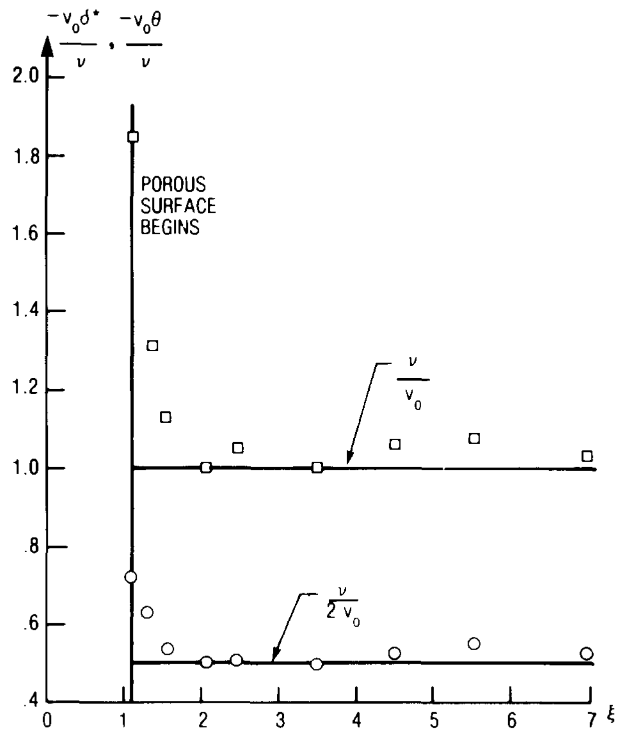

Figure 12. Approach to the asymptotic condition of two-dimensional laminar flow over a flat plate with uniform suction – data of J.M. Kay [12].

displacement thickness at 8 different locations on the plate between the start and end of the porous section. In addition, establishment of the Blasius profile at the end of the leading non-porous section was demonstrated.

Plots of the momentum and displacement thickness for this test series are given in dimensionless form in Figure 12 as a function of distance from the leading edge. According to J.M.

Kay, 'With suction applied, the velocity profiles approach rapidly towards the laminar asymptotic exponential form . . . as the distance X along the plate increases. This process is revealed clearly in (Figure 12) where the displacement thickness δ^* and the momentum thickness θ are plotted against distance X (dimensionless coordinate ξ). From $X = 8$ in. onwards, the momentum thickness remains substantially constant and is in close agreement with the theoretical asymptotic value $\nu/2v_0$. The displacement thickness settles down in a similar manner to the theoretical asymptotic value ν/v_0. There is some slight variation from point to point along the plate but this is simply due to local variation in the porosity of the surface. For example, at the point corresponding to the velocity profile of $X = 21.7$ in. ($\xi = 5.5$) the porosity is below standard but further on as at $X = 27.4$ in. ($\xi = 7.0$) the surface is better and the profile is in closer agreement with the asymptotic form.'

Application of the initial-length requirement based on Iglisch's analysis would predict attainment of the asymptotic state 15.72 inches from the beginning of the porous wall, while J.M. Kay observed that less than 4 inches was required.

A second pertinent series of experiments were conducted by M.R. Head [13], utilizing an airfoil section mounted vertically under the fuselage centerline of a twin-engine air-fuselage plane. The porous surface of the airfoil extended from approximately 10 to 68 percent of chord. Many sets of boundary-layer traverse data are presented, with varying suction rates. Acceptable agreement with the exponential profile is found with ξ as low as 0.608.

M.R. Head presents one more set of data of interest pertinent to the asymptotic theory. With regard to the variation of skin friction along the centerline of the aerofoil, the author states: 'It will be observed that, for $v_0/U_0 = .0016$, the distribution of skin friction is somewhat similar to that of suction velocity. This curve also indicates that, had the suction velocity been uniform (the) asymptotic condition would have been rapidly approached.' Study of the data reveals that for a choice of $\xi = 4.0$ the initial length required would be on the order of 14 inches while the skin friction data indicates establishment of the exponential profile within 2 inches of the start of the porous surface.

Later analysis appears to support the conclusion that an initial length derived from Iglisch's calculations is excessively long. A summary of theoretical investigation regarding development of the asymptotic profile after a solid length, reported in [6], indicates a change in ξ of less than 0.5 may be sufficient in characterizing the approach to the asymptotic state.

To summarize, the initial-length requirement for attainment of the asymptotic state is probably considerably less than estimated on the basis of Iglisch's analysis. This is encouraging with regard to application of the asymptotic theory in two or three dimensions.

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